

Free energy of the three-state $\tau_2(t_q)$ model as a product of elliptic functions

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Abstract

We show that the free energy of the three-state $\tau_2(t_q)$ model can be expressed as products of Jacobi elliptic functions, the arguments being those of an hyperelliptic parametrization of the associated chiral Potts model. This is the first application of such a parametrization to the N -state chiral Potts free energy problem for $N > 2$.

Keywords: Statistical mechanics, lattice models, free energy, chiral Potts model, τ_2 model

1. Introduction

In the field of solvable models in statistical mechanics, the N -state two-dimensional chiral Potts model has proved particularly challenging. This is despite the fact that for $N = 2$ it reduces to the Ising model, the free energy of which was calculated by Onsager (1944).

The model was developed by Howes, Kadanoff and den Nijs (1983), von Gehlen and Rittenberg (1985), Au-Yang *et al* (1987) and McCoy, Perk and Tang (1987). It was first fully defined as a general N -state lattice model by Baxter, Perk & Au-Yang (1988), who showed that it satisfied the star-triangle relations. These relations define “rapidity” variables p, q such that the Boltzmann weights are functions W_{pq} of p and q .

Many of the citations herein are to earlier papers by the author: in these we shall abbreviate “Baxter” to “B”.

With previously solved models, such as the hard-hexagon model (B 1980), the calculation of the free energy was straightforward once one had obtained the star-triangle relation and rapidities for that model. Typically, it turned out that there was a transformation in terms of Jacobi elliptic functions such that W_{pq} depended on p, q only via their difference $u = q - p$. The free energy is $-k_B T \log \kappa_{pq}$, where κ_{pq} is the partition function per site. This κ_{pq} must

also be a function $\kappa(u)$ of u , and there were always “rotation” and “inversion” relations (B 1982a, 1982b), of the form

$$\begin{aligned}\kappa(u) &= \kappa(\lambda - u) \\ \kappa(u) \kappa(-u) &= r(u) \ ,\end{aligned}\tag{1.1}$$

where λ is a positive real constant (the “crossing parameter”) and $r(u)$ a known meromorphic function. For u real and $0 < u < \lambda$ the Boltzmann weights are real and positive, and one can develop low-temperature series expansions in the usual way that indicate that $\kappa(u)$ is analytic, non-zero and bounded in the vertical strip $0 \leq \Re(u) \leq \lambda$. The equations (1.1) then determine $\kappa(u)$ and one can solve them by Fourier transforms or other methods.

The $N > 2$ chiral Potts model, however, does *not* have the “rapidity difference property”, i.e. there is no transformation that takes W_{pq} to a function only of $q - p$. This makes the calculation of κ_{pq} much more difficult, and it was not till 1990 that an explicit result was obtained (B 1990, 1991a). The calculation of the spontaneous magnetizations (order parameters) is even harder, and was not accomplished until 2005 (B 2005a, 2005b).

The calculation of κ_{pq} proceeds in two stages. First one calculates the partition function per site $\tau_2(p, q)$ of an associated “ $\tau_2(t_q)$ model”, which is closely related to the superintegrable case of the chiral Potts model. Then one uses this result to calculate κ_{pq} .

In the calculation of the order parameters, we used the fact that certain functions $G_{p,V_p}(r)$, $\mathcal{S}(p)$ are similar to $\tau_2(p, q)$. In fact they are ratios of special cases of $\tau_2(p, q)$.

Although the main task, namely the calculation of the free energy and order parameters of the chiral Potts model, has been accomplished, it remains disappointing that one has no elegant parametrization in terms of elliptic functions, as one has for models with the difference property. These parametrizations explicitly exhibit the poles and zeros of κ_{pq} on an extended p, q Riemann surface. There is a parametrization of the rapidities and Boltzmann weights of the N -state chiral Potts model in terms of hyperelliptic functions (B 1991b), but these have $N - 1$ arguments that are related to one another in a complicated way, and until recently they have not been found to be particularly useful.

However, we have now found that for $N = 3$ the function $\mathcal{S}(p)$ mentioned above can be expressed as a ratio of generalized elliptic functions of these arguments (B 2006). The obvious question is whether $\tau_2(p, q)$ can be similarly expressed. We show here that the answer is yes. The functions that occur are the same as that appear in other solvable models, notably the Ising model.

2. The function $\tau_2(p, q)$

For the chiral Potts model, the rapidity p can be thought of as the set of variables $p = \{x_p, y_p, \mu_p, t_p\}$, related to one another by

$$t_p = x_p y_p \ , \quad x_p^N + y_p^N = k(1 + x_p^N y_p^N) \ ,$$

(2.1)

$$kx_p^N = 1 - k'/\mu_p^N \quad , \quad ky_p^N = 1 - k'\mu_p^N \quad .$$

Here k, k' are real positive constants, satisfying

$$k^2 + k'^2 = 1 \quad . \quad (2.2)$$

In terms of the a_p, b_p, c_p, d_p of (Baxter, Perk & Au-Yang 1988; B 1991b), $x_p = a_p/d_p$, $y_p = b_p/c_p$, $\mu_p = d_p/c_p$.

There are various automorphisms or maps that take one set $\{x_p, y_p, \mu_p, t_p\}$ to another set satisfying the same relations (2.1). Four that we shall use are:

$$\begin{aligned} R : \{x_{Rp}, y_{Rp}, \mu_{Rp}, t_{Rp}\} &= \{y_p, \omega x_p, 1/\mu_p, \omega t_p\} \quad , \\ U : \{x_{Up}, y_{Up}, \mu_{Up}, t_{Up}\} &= \{x_p^{-1}, y_p^{-1}, \omega^{-1/2} x_p \mu_p / y_p, t_p^{-1}\} \quad , \\ V : \{x_{Vp}, y_{Vp}, \mu_{Vp}, t_{Vp}\} &= \{x_p, \omega y_p, \mu_p, \omega t_p\} \quad , \\ M : \{x_{Mp}, y_{Mp}, \mu_{Mp}, t_{Mp}\} &= \{x_p, y_p, \omega \mu_p, t_p\} \quad , \end{aligned} \quad (2.3)$$

where

$$\omega = e^{2\pi i/N} \quad .$$

They satisfy

$$\begin{aligned} RV^{-1}R &= V \quad , \quad MRM = R \quad , \\ U^2M &= V^N = M^N = 1 \quad . \end{aligned} \quad (2.4)$$

Here $U = RSV$, S being the operator S used in (Baxter, Perk & Au-Yang 1988, B 2006).

We take μ_p to be outside the unit circle, so

$$|\mu_p| > 1 \quad . \quad (2.5)$$

Then we can specify x_p uniquely by requiring that

$$-\pi/(2N) < \arg(x_p) < \pi/(2N) \quad . \quad (2.6)$$

We regard x_p, y_p, μ_p^N as functions of t_p . Then t_p lies in a complex plane containing N branch cuts $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{N-1}$ on the lines $\arg(t_p) = 0, 2\pi/N, \dots, 2\pi(N-1)/N$, as indicated in Fig. 1, while x_p lies in a near-circular region round the point $x_p = 1$, as indicated schematically by the region \mathcal{R}_0 inside the dotted curve of Fig. 1. The variable y_p can lie anywhere in the complex plane *except* in \mathcal{R}_0 and in $N-1$ corresponding near-circular regions $\mathcal{R}_1, \dots, \mathcal{R}_{N-1}$ round the other branch cuts. With these choices, we say that p lies in the “domain” \mathcal{D} .

It can be helpful to consider the low-temperature limit, when $k' \rightarrow 0$. The branch cuts $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{N-1}$ and the regions $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{N-1}$ then shrink to the points $1, \omega, \dots, \omega^{N-1}$. If t_p is held fixed, not at $1, \omega, \dots, \omega^{N-1}$, then $x_p \rightarrow 1, y_p \rightarrow t_p$ and $\mu_p^N = O(1/k')$.

Define q similarly to p , also lying in \mathcal{D} . Then from eq. 39 of (B 1991a) and eq. 54 of (B 2003a), the partition function per site $\tau_2(p, q) = \tau_2(\mu_p, t_q)$ of the $\tau_2(t_q)$ model is given by

$$\log \tau_2(p, q) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 + e^{i\theta}/\mu_p^N}{1 - e^{i\theta}/\mu_p^N} \right) \log [\Delta(\theta) - \omega t_q] \, d\theta \quad , \quad (2.7)$$

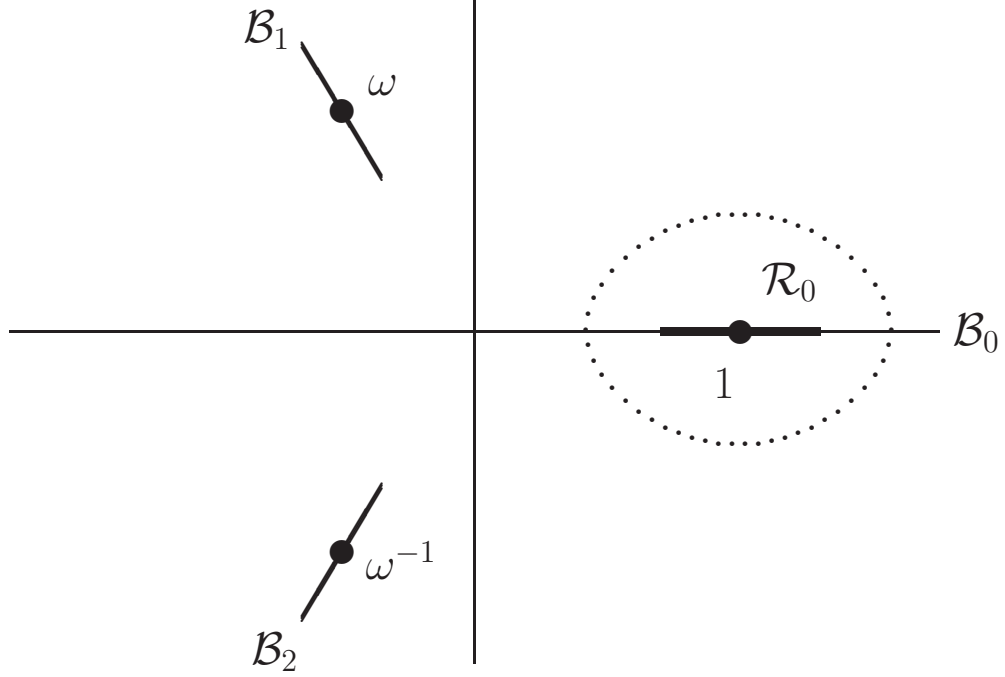


Figure 1: The cut t_p -plane for $N = 3$, showing the three branch cuts $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ and the approximately circular region \mathcal{R}_0 in which x_p lies when $p \in \mathcal{D}$.

where

$$\Delta(\theta) = [(1 - 2k' \cos \theta + k'^2)/k^2]^{1/N} . \quad (2.8)$$

We have omitted a factor y_p^2 from $\tau_2(p, q)$, and to ensure that this formula is correct for $p, q \in \mathcal{D}$, we invert μ_p, μ_q in (B 1991a, 2003a).

This function $\tau_2(p, q)$ satisfies the relations

$$\tau_2(Vp, q) = \tau_2(p, q) , \quad (2.9)$$

and

$$\prod_{j=0}^{N-1} \tau_2(p, V^j q) = \prod_{j=0}^{N-1} \tau_2(\mu_p, \omega^j t_q) = \alpha(p, q) , \quad (2.10)$$

where

$$\alpha(p, q) = \frac{k'(\mu_q^N - \mu_p^{-N})^2}{k^2 \mu_q^N} = \frac{(x_p^N - y_q^N)(x_q^N - y_p^N)}{k' \mu_p^N} . \quad (2.11)$$

3. The Riemann sheets (“domains”) formed by analytic continuation

We shall want to consider the analytic continuation of certain functions of t_p onto other Riemann sheets, i.e. beyond the domain \mathcal{D} . We restrict

attention to functions that are meromorphic and single-valued in the cut plane of Figure 1, and similarly for their analytic continuations. Obvious examples are x_p, y_p and $\tau_2(t_p)$. They are therefore meromorphic and single-valued on their Riemann surfaces, but we need to know what these surfaces are.

We start by considering the most general such surface. As a first step, allow μ_p to move from outside the unit circle to inside. Then t_p will cross one of the N branch cuts \mathcal{B}_i in Figure 1, moving onto another Riemann sheet, going back to its original value but now with y_p in \mathcal{R}_i . Since y_p is thereby confined to the region near and surrounding ω^i , we say that $y_p \simeq \omega^i$. Conversely, by $y_p \simeq \omega^i$ we mean that $y_p \in \mathcal{R}_i$.

We say that p has moved into the *domain* \mathcal{D}_i *adjacent* to \mathcal{D} . There are N such domains $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{N-1}$.

Now allow μ_p to become larger than one, so t_p again crosses one of the N branch cuts. Again we require that t_p returns to its original value. If it crosses \mathcal{B}_i , then it moves back to the original domain \mathcal{D} . However, if it crosses another cut \mathcal{B}_j then x_p moves into \mathcal{R}_{j-i} , and we say that p is now in domain $\mathcal{D}_{i,j-i}$.

Proceeding in this way, we build up a Cayley tree of domains. For instance, the domain \mathcal{D}_{ijk} is a third neighbour of \mathcal{D} , linked via the first neighbour \mathcal{D}_i and the second-neighbour \mathcal{D}_{ij} , as indicated in Figure 2. Here $x_p \simeq 1$ in \mathcal{D} , $y_p \simeq \omega^i$ in \mathcal{D}_i , $x_p \simeq \omega^j$ in \mathcal{D}_{ij} and $y_p \simeq \omega^k$ in \mathcal{D}_{ijk} . We reject moves that take p back to the domain immediately before the last, so $j \neq 0$ and $k \neq i$. We refer to the sequence $\{i, j, k, \dots\}$ that define any domain as a *route*. We can think of it as a sequence of points, all with the same value of t_p , on the successive Reimann sheets or domains.

The domains $\mathcal{D}, \mathcal{D}_{ij}, \mathcal{D}_{ijk\ell}, \dots$ with an even number of indices, have $x \simeq \omega^\ell$, where ℓ is the last index. We refer to them as being of even *parity* and of *type* ℓ . The domains $\mathcal{D}_i, \mathcal{D}_{ijk}, \dots$ have $y \simeq \omega^\ell$ and are of odd parity and type ℓ .

$$\mathcal{D} \text{ --- } \mathcal{D}_i \text{ --- } \mathcal{D}_{ij} \text{ --- } \mathcal{D}_{ijk}$$

Figure 2: A sequence of adjacent domains $\mathcal{D}, \mathcal{D}_i, \mathcal{D}_{ij}, \mathcal{D}_{ijk}$.

The automorphism that takes a point p in \mathcal{D} to a point in \mathcal{D}_i , respectively, is the mapping

$$A_i = V^{i-1} R V^{-i} . \quad (3.1)$$

If $p' = A_i p$, then

$$x_{p'} = \omega^{-i} y_p , \quad y_{p'} = \omega^i x_p , \quad t_{p'} = t_p . \quad (3.2)$$

Because of (2.4), $A_{i+N} = A_i$, so there are N such automorphisms.

We can use these maps to generate all the sheets in the full Cayley tree. Suppose we have a domain with route $\{i, j, k, \dots\}$ and we apply the automorphism A_α to all points on the route. From (3.2) this will generate a new route $\{\alpha, i - \alpha, j + \alpha, k - \alpha, \dots\}$. For instance, if we apply the map A_α to the route $\{m\}$ from \mathcal{D} to \mathcal{D}_m , we obtain the route $\{\alpha, m - \alpha\}$ to the domain $D_{\alpha, m - \alpha}$. Thus the map that takes \mathcal{D} to \mathcal{D}_{ij} is $A_i A_{i+j}$.

Iterating, we find that the map that takes \mathcal{D} to $\mathcal{D}_{ijk\dots mn}$ is

$$A_i A_{i+j} A_{j+k} \cdots A_{m+n} . \quad (3.3)$$

We must have

$$A_i^2 = 1 , \quad (3.4)$$

since applying the same map twice merely returns p to the previous domain.

Let us refer to the general Riemann surface we have just described as \mathcal{G} . It consists of infinitely many Riemann sheets, each sheet corresponding to a site on a Cayley tree, adjacent sheets corresponding to adjacent points on the tree. A Cayley tree is a huge graph: it contains no circuits and is infinitely dimensional, needing infinitely many integers to specify all its sites.

Any given function will have a Riemann surface that can be obtained from \mathcal{G} by identifying certain sites with one another, thereby creating circuits and usually reducing the graph to one of finite dimensionality.

From (3.2), the maps A_0, A_1, \dots, A_{N-1} leave t_p unchanged. We shall often find it helpful to regard t_p as a fixed complex number, the same in all domains, and to consider the corresponding values of x_p, y_p (and the hyperelliptic variables z_p, w_p) in the various domains. To within factors of ω , the variables x_p and y_p will be the same as those for \mathcal{D} in even domains, while they will be interchanged on odd domains.

(a) Analytic continuation of $\tau_2(p, q)$

Consider $\tau_2(p, q)$ as a function of q (so replace p by q in the previous discussion of the domains). More specifically, think of it as a function of the complex variable t_q . For $q \in \mathcal{D}$ (and $p \in \mathcal{D}$), it is apparent from (2.7) that $\tau_2(p, q)$ is an analytic function of t_q except for the *single* branch cut \mathcal{B}_{N-1} , being single-valued across the other $N - 1$ cuts. Let $q' = A_i q$, so $q' \in \mathcal{D}_i$, and define

$$\begin{aligned} \delta(i, j) &= 1 && \text{if } i = j, \text{ mod } N, \\ \delta(i, j) &= 0 && \text{else.} \end{aligned} \quad (3.5)$$

Then it follows from (2.10) that

$$\tau_2(p, q') = v_{pq}^{-\delta(i, N-1)} \tau_2(p, q) \quad (3.6)$$

for $i = 0, 1, \dots, N - 1$, where

$$v_{pq} = \frac{\alpha(p, q)}{\alpha(p, q')} = \frac{(x_p^N - y_q^N)(y_p^N - x_q^N)}{(x_p^N - x_q^N)(y_p^N - y_q^N)} = 1/v_{pq'} \quad (3.7)$$

and $x_{q'}^N = y_q^N, y_{q'}^N = x_q^N$.

Eqn. (3.6) defines the mapping A_i applied to the function $\tau_2(p, q)$ of q . Iterating, it follows that if $q'' = A_i A_j A_k q$, then

$$\tau_2(p, q'') = v_{pq}^{-m} \tau_2(p, q) , \quad (3.8)$$

where $m = \delta(i, N-1) - \delta(j, N-1) + \delta(k, N-1)$.

We can also keep q fixed in \mathcal{D} and consider the analytic continuation of $\tau_2(p, q)$ as p moves from sheet to sheet. If $p' = A_i p$, so $p \in \mathcal{D}$, $p' \in \mathcal{D}_i$, we can verify from (2.7) that

$$\tau_2(p', q) = (\omega^{-i} t_p - \omega t_q)^2 / \tau_2(p, q) , \quad t_{p'} = t_p . \quad (3.9)$$

Also, from (2.11),

$$\alpha(p, q) \alpha(p', q) = (t_p^N - t_q^N)^2 . \quad (3.10)$$

Eqn. (3.9) defines the A_i for the function $\tau_2(p, q)$ of p . Iterating, it follows that if $p'' = A_i A_j A_k p$, then

$$\tau_2(p'', q) = (\omega^{-i} t_p - \omega t_q)^2 (\omega^{-j} t_p - \omega t_q)^{-2} (\omega^{-k} t_p - \omega t_q)^2 / \tau_2(p, q) . \quad (3.11)$$

Note that for both $\tau_2(p, q) \rightarrow \tau_2(p, q'')$ and $\tau_2(p, q) \rightarrow \tau_2(p'', q)$, it is true that $A_i A_j A_k = A_k A_j A_i$.

4. Hyperelliptic parametrization for $N = 3$

Hereinafter we restrict our attention to the case $N = 3$. A parametrization of $\{x_p, y_p, \mu_p, t_p\}$ was developed in previous papers, in terms of a “nome” x and two related parameters z_p, w_p (B 1991b, 1993a, 1993b, 1998). The nome x is like k and k' in that it is a constant: it is not to be confused with the rapidity variable x_p .

However, in (B 2006) we showed that the function \mathcal{S}_p could not be expressed as a single-valued function of these original parameters z_p, w_p . This is because z_p, w_p have the same values (for given t_p) in the domains \mathcal{D}_{021} and \mathcal{D}_{211} , whereas \mathcal{S}_p has different values therein. These domains are obtained by the maps $A_0 A_2 A_0$, $A_2 A_0 A_2$, respectively, and indeed we see from (3.8) and (3.9) that these maps give different results for $\tau_2(p, q)$ for both the q and the p variables. Thus $\tau_2(p, q)$ cannot be a single-valued function, either of z_q, w_q for fixed p , or of z_p, w_p for fixed q .

The same problem occurs with the domains \mathcal{D}_{110} , \mathcal{D}_{220} and the corresponding maps $A_1 A_2 A_1$, $A_2 A_1 A_2$.

The situation is not lost. We also showed in (B 2006) that there is another way of parametrizing k, x_p, y_p, μ_p, t_p . This alternative way preserves the property that the nome x is small at low temperatures (k' small). It can be obtained from the original parametrization by leaving x, z_p, w_p unchanged and transforming $k, k', x_p, y_p, \mu_p, t_p$ according to the rule:

$$k, k', x_p, y_p, \mu_p, t_p \rightarrow k^{-1}, i k' / k, 1 / x_p, y_p, \omega^{-1/4} x_p \mu_p, y_p / x_p \quad (4.1)$$

(taking $\omega^{-1/4} = e^{-i\pi/2N}$). This mapping leaves the relations (2.1), (2.2) unchanged.

Doing this, eqn. (21) of (B 1993a) becomes

$$-k'^2 = 27x \prod_{n=1}^{\infty} \left(\frac{1-x^{3n}}{1-x^n} \right)^{12} , \quad (4.2)$$

while the two equivalent relations (4.5), (4.6) of (B 1993b) remain unchanged:

$$\begin{aligned} w &= \prod_{n=1}^{\infty} \frac{(1-x^{2n-1}z/w)(1-x^{2n-1}w/z)(1-x^{6n-5}zw)(1-x^{6n-1}z^{-1}w^{-1})}{(1-x^{2n-2}z/w)(1-x^{2n}w/z)(1-x^{6n-2}zw)(1-x^{6n-4}z^{-1}w^{-1})} \\ \frac{z}{w} &= \prod_{n=1}^{\infty} \frac{(1-x^{2n-2}/w)(1-x^{2n}w)(1-x^{6n-4}z^2/w)(1-x^{6n-2}w/z^2)}{(1-x^{2n-1}w)(1-x^{2n-1}/w)(1-x^{6n-5}w/z^2)(1-x^{6n-1}z^2/w)} . \end{aligned} \quad (4.3)$$

These z, w are related to x_p, y_p, μ_p, t_p by various elliptic-type equations that we shall give below, being the arguments (more precisely the exponentials of 1 times the arguments) of Jacobi elliptic functions of nome x . Thus x , like k, k' , is a constant, while z, w are two more rapidity variables, dependent on p . We shall them as z_p, w_p .

First we introduce elliptic-type functions

$$h(z) = 1/h(z^{-1}) = \omega^2 h(xz) = -\omega^2 \prod_{n=1}^{\infty} \frac{(1-\omega x^{n-1}z)(1-\omega^2 x^n/z)}{(1-\omega^2 x^{n-1}z)(1-\omega x^n/z)} ,$$

$$\phi_b(z) = \prod_{n=1}^{\infty} \frac{(1-x^{3n-2}/z)(1-x^{3n-1}z)}{(1-x^{3n-2}z)(1-x^{3n-1}/z)} ,$$

$$\phi(z) = 1/\phi(z^{-1}) = \phi(x^3 z) = z^{1/3} \phi_b(z) \quad (4.4)$$

so that $h(1) = \phi(1) = 1$, and define two sets of parameters p, \bar{p} in terms of z_p, w_p :

$$p_1 = z_p , \quad p_2 = -1/w_p , \quad p_3 = -w_p/z_p , \quad (4.5)$$

$$\bar{p}_j = p_{j+1}/p_{j-1} ,$$

extending p_j, \bar{p}_j to all j by

$$p_{j+3} = p_j , \quad \bar{p}_{j+3} = \bar{p}_j . \quad (4.6)$$

Thus

$$\bar{p}_1 = z_p/w_p^2 , \quad \bar{p}_2 = -w_p/z_p^2 , \quad \bar{p}_3 = -z_p w_p . \quad (4.7)$$

We shall also need certain cube roots \hat{p}_j of $x \bar{p}_j$, choosing them so that

$$\hat{p}_j = (x \bar{p}_j)^{1/3} = \omega p_{j+1} \hat{p}_{j-1} . \quad (4.8)$$

Then after applying the mapping (4.1), the equations (27), (32) of (B 1993a) become

$$y_p/x_p = -\omega^{-j} h(p_j) , \quad (4.9)$$

$$\frac{y_p}{x_p \mu_p^2} = -\omega^{-j} \phi(x \bar{p}_j) = -\omega^{-j} \hat{p}_j \phi_b(x \bar{p}_j) , \quad (4.10)$$

for $j = 1, 2, 3$. The second form of (4.10) is to be preferred as it fixes the choice of the leading cube root factor in the function $\phi(z)$, so fixing \hat{p}_j .

Similarly, replacing p by q in the above equations, we define $q_j, \bar{q}_j, \hat{q}_j$, and hence x_q, y_q, μ_q, t_q , in terms of two related variables z_q, w_q .

(a) *The low-temperature limit*

At low temperatures x, k' are small. For $p \in \mathcal{D}$ (and t_p not close to a cube root of unity) we can choose z_p, w_p to tend to non-zero limits as $x \rightarrow 0$. Then (4.3) both give $w_p = z_p + 1$. Also, $x_p \rightarrow 1$, so (2.1) and (4.9) determine x_p, y_p uniquely (with the same value for $j = 1, 2, 3$). Then μ_p^3 can be calculated from the last of the eqns. (2.1), and μ_p itself from (4.10). Hence when x, k' are small

$$\begin{aligned} k'^2 &= -27x \quad , \quad w_p = z_p + 1 \quad , \quad x_p = 1 \quad , \\ y_p &= \frac{\omega^2 - z_p}{\omega - z_p} \quad , \quad k' \mu_p = -3(1 - \omega^j y_p) \hat{p}_j \quad , \end{aligned}$$

for $j = 1, 2, 3$. Note that $x_p, y_p, z_p, w_p, p_j, \bar{p}_j$ are of order unity, but the \hat{p}_j are of order $x^{1/3}$, while μ_p is of order $x^{-1/6}$.

(b) *Mappings*

The effect of the automorphisms R, U, V on z_p, w_p is

$$\begin{aligned} z_{Rp} &= -xw_p \quad , \quad z_{Up} = -w_p \quad , \quad z_{Vp} = -1/w_p \quad , \\ w_{Rp} &= w_p/z_p \quad , \quad w_{Up} = -z_p \quad , \quad w_{Vp} = z_p/w_p \quad . \end{aligned} \quad (4.11)$$

The mappings U, V, M take a point p within \mathcal{D} to another point within \mathcal{D} :

$$\begin{aligned} p' = Up : \quad p'_j &= 1/p_{3-j}, \quad \bar{p}'_j = \bar{p}_{-j}, \quad \hat{p}'_j = \omega^{1-j} \hat{p}_{-j} \quad , \\ p' = Vp : \quad p'_j &= p_{j+1}, \quad \bar{p}'_j = \bar{p}_{j+1}, \quad \hat{p}'_j = \hat{p}_{j+1} \quad , \\ p' = Mp : \quad p'_j &= p_j \quad , \quad \bar{p}'_j = \bar{p}_j \quad , \quad \hat{p}'_j = \omega \hat{p}_j \quad . \end{aligned} \quad (4.12)$$

and if $p' = Rp$, then $\hat{p}'_j = \omega^{2-j} x^{1-\delta(j,2)} / \hat{p}_{j+1}$.

Also, if $p' = A_i p$, then

$$\begin{aligned} z_{p'} &= x^{2-i-3\delta(i,0)} / z_p \quad , \quad w_{p'} = x^{i-1} / w_p \\ p'_j &= x^{\delta(i+j,2)-\delta(i+j,1)} / p_j \quad , \quad (\hat{p}_j)' = \omega^{-i-j} x^{1-\delta(i+j,0)} / \hat{p}_j \quad , \end{aligned} \quad (4.13)$$

for $i = 0, 1, 2$. We see that in this new parametrization $A_0 A_2 A_0$ does *not* have the same effect on z_p, w_p as $A_2 A_0 A_2$ (nor do $A_1 A_2 A_1$ and $A_2 A_1 A_2$), so we no longer have the problem referred to at the beginning of this section.

If we identify Riemann sheets that have the same values of z_p, w_p for given t_p , then \mathcal{G} ceases to be a Cayley tree and becomes the two-dimensional honeycomb lattice of Figure 3.

To see this, note that if z_p^0, w_p^0 are the values of z_p, w_p on the central sheet \mathcal{D} , then it follows from (4.13) that on any Riemann sheet the analytic continuations of z_p, w_p (for a given value of t_p) are

$$z_p = x^m \left(z_p^0 \right)^{\pm 1} \quad , \quad w_p = x^n \left(w_p^0 \right)^{\pm 1} \quad , \quad (4.14)$$

We noted in the previous section that the function $\tau_2(p, q)$, considered as a function of either p or q , also satisfies (4.16). It follows that its Riemann surface can be embedded in that of the reduced graph \mathcal{G} .

We focus on the q dependence of $\tau_2(p, q)$. Let its value when $p, q \in \mathcal{D}$ (given by eq. 2.7) be $\tau_2^0(p, q)$. Then by iterating (3.6) we find that on the Riemann sheet (m, n)

$$\tau_2(p, q) = v_{pq}^r \tau_2^0(p, q) , \quad (4.17)$$

where

$$\begin{aligned} r &= (m - 2n)/3 \quad \text{on even sheets,} \\ r &= (m - 2n - 1)/3 \quad \text{on odd sheets.} \end{aligned} \quad (4.18)$$

(c) The function $T(p, q)$

We can eliminate the distinction between even and odd sheets by working, not with $\tau_2(p, q)$, but instead with the closely related function

$$T(p, q) = \frac{\tau_2(p, q)^3}{\alpha(p, q)} = \frac{\tau_2(p, q)^2}{\tau_2(p, Vq) \tau_2(p, V^2q)} , \quad (4.19)$$

(using eqn. 2.10). Then (3.6) becomes

$$T(p, A_i q) = v_{pq}^{1-3\delta(i, N-1)} T(p, q) \quad (4.20)$$

for $i = 0, 1, 2$. If $T_0(p, q)$ is the value of $T(p, q)$ in \mathcal{D} , then it follows that for a given value of t_q ,

$$T(p, q) = v_{pq}^{m-2n} T_0(p, q) , \quad (4.21)$$

on all sheets (m, n) .

It follows that $T(p, q)$ is a single-valued meromorphic function on the Riemann surface \mathcal{G} , and that the orders of its zeros and poles are linear in the integers m, n of (4.14) (with p replaced by q therein). This suggests that it may be possible to write $T(p, q)$ as a product of functions of z_q, w_q , and indeed we shall find that this is the case.

We shall explicitly exhibit the dependence of $T(p, q)$ on z_q, w_q by writing it as $T_p(z_q, w_q)$. Then the three relations (4.20) become

$$\begin{aligned} T_p(x^{-1}/z_q, x^{-1}/w_q) &= T_p(x/z_q, 1/w_q) = v_{pq} T_p(z_q, w_q) , \\ T_p(1/z_q, x/w_q) &= v_{pq}^{-2} T_p(z_q, w_q) . \end{aligned} \quad (4.22)$$

If p is fixed within \mathcal{D} , then from (2.7) and (4.19), $T_p(z_q, w_q)$ is bounded and has no zeros or poles for $q \in \mathcal{D}$. (Both τ_2^3 and α become infinite as $y_q, t_q \rightarrow \infty$, but their ratio remains finite.) Together with this restriction, the relations (4.22) define $T_p(z_q, w_q)$ to within a multiplicative factor independent of q . To see this, suppose we had two such solutions, then (4.22) would imply that their ratio was unchanged by $q \rightarrow A_i q$, for $i = 0, 1, 2$. This would mean that the ratio, considered as a functions of t_q , did not have the branch cuts $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$. It would therefore be an entire bounded function of t_q , and hence

by Liouville's theorem a constant (independent of q). This constant could be determined from the product relation (2.10), which now takes the simple form

$$T_p(z_q, w_q)T_p(-1/w_q, z_q/w_q)T_p(-w_q/z_q, -1/z_q) = 1 . \quad (4.23)$$

We shall also use the p -relation (3.9). Together with (3.10) this implies that

$$T(p, q)T(A_i p, q) = \frac{(t_p - \omega^{i+1}t_q)^6}{(t_p^3 - t_q^3)^2} . \quad (4.24)$$

5. Various p, q relations

First we present various relations that enable one to express certain rational functions (including v_{pq}) of x_p, \dots, t_q as products of elliptic functions of z_p, w_p, z_q, w_q .

Some can be obtained by applying the rule (4.1) to eqn 34 of (B 1993a) or to eqn 4.9 of (B 1993b). In particular, we obtain

$$\frac{\mu_q(1 - x_q y_p)}{\mu_p(1 - x_p y_q)} = \phi(z_q/z_p)\phi(w_q/w_p) . \quad (5.1)$$

Applying the automorphism U to the q variable in (5.1), and then using the V and R automorphisms, we obtain the two sets of relations

$$\begin{aligned} \frac{\mu_p(y_q - \omega^j x_p)}{\mu_q(y_p - \omega^i x_q)} &= \omega^{j-i} \frac{\phi(x^m p_{m-i+1} q_{m-j-1})}{\phi(x^m p_{m-i-1} q_{m-j+1})} , \\ \frac{\mu_p \mu_q (x_q - \omega^j x_p)}{y_p - \omega^i y_q} &= \omega^{m-i} \frac{\phi(x^{j+1} p_{m-i+1}/q_{m-j-1})}{\phi(x^{j-1} p_{m-i-1}/q_{m-j+1})} , \end{aligned} \quad (5.2)$$

true for all integers i, j, m .

The function $\phi(z)$ has a leading factor $z^{1/3}$, which gives a contribution $z_q w_q / (z_p w_p)^{1/3}$ to the RHS of (5.1). This cube root can and should be chosen to be \hat{q}_3/\hat{p}_3 . The corresponding contributions in (5.2) are $\hat{p}_{m-i}/\hat{q}_{m-j}$ and $\hat{p}_{m-i}\hat{q}_{m-j}$.

We define the elliptic function

$$g_r(z) = \prod_{n=1}^{\infty} (1 - x^{rn-r}z)(1 - x^{rn}/z) \quad (5.3)$$

for integer r (in particular $r = 1$ and 3), satisfying

$$g_r(x^r z) = g_r(1/z) = -z^{-1} g_r(z) .$$

Then $h(z) = \omega^2 g_1(\omega z)/g_1(\omega^2 z)$ and $\phi_b(z) = g_3(x/z)/g_3(xz)$, so (4.9), (4.10) can be used to relate various g_1 and g_3 functions. In particular, from (4.8) and (4.10),

$$g_3(\bar{p}_j)/g_3(\bar{p}_i) = \omega^2 p_k^2 g_3(x^2 \bar{p}_j)/g_3(x^2 \bar{p}_i) , \quad (5.4)$$

for any cyclic permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. We note that (4.2) can be written

$$-k'^2 = 27x/g_3(x)^{12} \quad (5.5)$$

and set

$$\begin{aligned}\gamma &= (-27xk^4/k'^2)^{1/6} = -3\omega k^{2/3}/g_1(\omega)^2 \\ &= k^{2/3}g_3(x)^2 = 1 + 7x + 8x^2 + 22x^3 + 42x^4 + \cdots .\end{aligned}\quad (5.6)$$

It follows from (5.3) and (5.5) that these various expressions for γ are all equal.

We also define

$$\begin{aligned}v_1(p, q) &= \frac{(y_p^3 - x_q^3)(y_q^3 - 1/y_p^3)}{(y_p^3 - y_q^3)(x_q^3 - 1/y_p^3)} , \\ v_3(p, q) &= \frac{(x_p^3 - y_q^3)(x_q^3 - 1/y_p^3)}{(x_p^3 - x_q^3)(y_q^3 - 1/y_p^3)} ,\end{aligned}\quad (5.7)$$

and note that

$$v_{pq} = v_1(p, q)v_3(p, q) . \quad (5.8)$$

We find the following six identities and indicate below the method of their proof. The first two can also be derived from (5.2) by applying the duality/conjugate modulus mapping of the Appendix.

We have also checked all the identities of this section numerically, for arbitrarily chosen x, z_p, z_q , to 25 digits of accuracy.

$$\begin{aligned}\frac{y_q - \omega^j x_p \mu_p \mu_q}{y_p - \omega^i x_q \mu_p \mu_q} &= \omega^{j-i} \frac{h(\omega^m \hat{p}_{m-i-1} \hat{q}_{m-j+1})}{h(\omega^m \hat{p}_{m-i+1} \hat{q}_{m-j-1})} , \\ \frac{x_q \mu_q - \omega^j x_p \mu_p}{y_p \mu_q - \omega^i y_q \mu_p} &= \omega^{m-i} \frac{h(\omega^{j-1} \hat{p}_{m-i-1} / \hat{q}_{m-j+1})}{h(\omega^{j+1} \hat{p}_{m-i+1} / \hat{q}_{m-j-1})} ,\end{aligned}\quad (5.9)$$

$$v_1(p, q) = \prod_{j=1}^3 \frac{g_1(z_q p_j)}{g_1(z_q / p_j)} , \quad (5.10)$$

$$v_3(p, q) = \frac{z_q^2 w_q^2}{x} \prod_{j=1}^3 \frac{g_3(-x^2 z_q w_q \bar{p}_j)}{g_3(-z_q w_q / \bar{p}_j)} , \quad (5.11)$$

$$\begin{aligned}\frac{\omega^2 g_1(p_i q_j) g_1(q_j / p_i)}{g_1(\omega^2 p_i) g_1(\omega^2 / p_i) g_1(\omega^2 q_j) g_1(\omega q_j)} &= \frac{g_3(\bar{q}_j / \bar{p}_i) g_3(x^2 \bar{p}_i \bar{q}_j)}{g_3(1 / \bar{p}_i) g_3(x / \bar{p}_i) g_3(\bar{q}_j) g_3(x^2 \bar{q}_j)} \\ &= \frac{\gamma(\omega^i t_p - \omega^j t_q)(\omega^{i+j} t_p t_q - 1)}{3 \omega^{i+j} t_p t_q} ,\end{aligned}\quad (5.12)$$

for $i, j, m = 1, 2, 3$.

Letting $q \rightarrow Vq$, we see that (5.10) remains true if z_q therein is replaced by $-1/w_q$ or by $-w_q/z_q$. Also, (5.11) is true if $-z_q w_q$ is replaced by z_q/w_q^2 or by $-w_q/z_q^2$.

(a) Proof of the identities

We have proved the identities (5.9) - (5.12). Let E_q be the ratio of the LHS to the RHS of any of these identities.

First note that *a priori* we expect E_q to be a single-valued function of t_q only if we introduce the branch cuts \mathcal{B}_i of Fig. 1 into the complex t_q -plane.

Indeed, the identities (5.9) involve μ_q and \hat{q}_j , and these will need additional cuts linking the \mathcal{B}_i in order to completely fix the choice of the cube root in (4.8).

The map that takes q from one such choice to another is the map $q \rightarrow Mq$. However, from (2.3) and (4.12), this merely increases (decreases) i, j, m by one in the first (second) set of the identities (5.9). If we take some symmetric function Φ_q of all the E_q (for all i, j, m), then Φ_q will be invariant under $q \rightarrow Mq$, which means that these extra cuts are not needed for this function.

The next step is to use (3.2) and (4.13) to show that the mapping $q \rightarrow A_i q$ merely permutes, and possibly inverts, the E_q of each set of identities. This is true for each of the sets of identities (5.10) - (5.12), but for (5.9) the two sets are interchanged by this mapping.

Again, let Φ_q be some symmetric function of all the E_q (and if necessary their inverses) within a set (e.g. the sum of the fifth powers of every E_q in one of the eqns. 5.12, summed over i and j), now regarding the two identities (5.9) as forming one combined set. Then Φ_q will be unchanged by each of the three mappings $q \rightarrow A_i q$, for $i = 1, 2, 3$. This means that it has the same value on either side of any of the branch cuts \mathcal{B}_i . The cuts are therefore unnecessary: Φ_q is a *single-valued function* of the complex variable t_q .

The only possible singularities of Φ_q are therefore poles, arising from poles, and possibly zeros, in the E_q . The only places these can occur are when $t_q^3 = t_p^3$, $t_q^3 = 1/t_p^3$, $t_q = 0$ and $t_q = \infty$. We can restrict our attention to $p, q \in \mathcal{D}$, since Φ_q is unchanged (for a given value of t_q) by crossing the \mathcal{B}_i . Thus $|\mu_q|$ and $|\mu_p|$ are both greater than one, and x_q, x_p each lie in a region near the points $x_q = 1$, $x_p = 1$. This means that the only points to consider are:

$$\begin{aligned}
1) \{x_q, y_q, \mu_q, q_j, \bar{q}_j, \hat{q}_j\} &= \{x_q, \omega^i y_q, \omega^m \mu_q, q_{j+i}, \bar{q}_{j+i}, \omega^m \hat{q}_{j+i}\} , \\
2) \{x_q, y_q, \mu_q, q_j, \bar{q}_j, \hat{q}_j\} &= \{1/x_q, \omega^i/y_q, \omega^{m-1/2} x_q \mu_q / y_q, \\
&\quad q_{-j-i}^{-1}, \bar{q}_{-j-i}, \omega^{m-j-1} \hat{q}_{-j-i}\} , \\
3) \{x_q, y_q, \mu_q, q_j, \bar{q}_j, \hat{q}_j\} &= \{k^{1/3}, 0, \omega^m/k'^{1/3}, \omega^2, 1, \omega^m x^{1/3}\} , \\
4) \{x_q, y_q, \mu_q, q_j, \bar{q}_j, \hat{q}_j\} &= \{k^{1/3}, \infty, \omega^m (-k/k')^{1/3} y_q, \omega, 1, \omega^{m-j} x^{1/3}\} ,
\end{aligned} \tag{5.13}$$

for all i and m .

Each E_q can be written as a ratio of pole-free functions. In every case, if the numerator (denominator) has a zero at one of the points (5.13), then so does the denominator (numerator), and both zeros are simple. Thus no E_q has a pole or zero at any of the above points.

The function Φ_q is therefore a single-valued and analytic function in the complete t_q plane, including the point at infinity (this is the last of the points listed above). By Liouville's theorem it is therefore a constant.

One can write down a polynomial of finite degree whose roots are the E_q and whose coefficients are symmetric functions Φ_q . Since each such coefficient

is a constant, so are the roots. Thus every E_q is a constant. By looking at special values of q , e.g. one of the values above, in each case we can show that the constant is unity, and hence prove the identities (5.9) - (5.12). In the last two identities one needs to take the limit as $y_q \rightarrow 0$ or ∞ , using (4.2), (4.9), (4.10) and (5.4) - (5.6).

(One can streamline the procedure: for instance, each of the 27 identities in the first set of relations (5.9) can be obtained from one of them by using the mappings $q \rightarrow Vq$, $q \rightarrow Mq$, $p \rightarrow Vp$, $p \rightarrow Mp$, so it sufficient to do the last step for just one of the equations, say $i = j = m = 0$.)

6. Calculation of $T(p, q)$

We now look for solutions of (4.22), using the identities (5.8) - (5.11). This leads us to define, similarly to section 6 of (B 2006), the function

$$F_r(z) = \prod_{j=1}^{\infty} \frac{(1 - x^{rj}z)^j}{(1 - x^{rj}z^{-1})^j} . \quad (6.1)$$

It satisfies

$$F_r(z) = 1/F_r(z^{-1}) = -z^{-1}g_r(z)F_r(x^r z) \quad (6.2)$$

and is a natural extension of the elliptic function $g_r(z)$.

We further define the functions

$$R_1(z) = \prod_{j=1}^3 F_1(z/p_j)/F_1(zp_j) ,$$

$$R_3(v) = \prod_{j=1}^3 F_3(v/\bar{p}_j)F_3(xv^{-1}/\bar{p}_j) ,$$

suppressing their dependence on p . They have been constructed so that

$$R_1(x^{-m}/q_i) = v_1(p, q)^m R_1(q_i) ,$$

$$R_3(x^{1-3m}/\bar{q}_i) = (-\bar{q}_i)^m x^{m(3m-1)/2} v_3(p, q)^m R_3(\bar{q}_i) , \quad (6.3)$$

for $i = 1, 2, 3$ and all integers m . Because they are products over p_1, p_2, p_3 (or $\bar{p}_1, \bar{p}_2, \bar{p}_3$), they, like $v_1(p, q)$, $v_3(p, q)$, $T(p, q)$, are unchanged by $p \rightarrow Vp$. They only have zeros or poles when $v_1(p, q)$, $v_3(p, q)$ have zeros or poles, i.e. when x_q^3 or y_q^3 equals $x_p^{\pm 3}$ or $y_p^{\pm 3}$. None of these zeros or poles occur when $p, q \in \mathcal{D}$.

The factors $(-\bar{q}_i)^m$, $x^{m(3m-1)/2}$ in (6.3) are irritating as they do not occur in (4.22). However, they are independent of p , so we may hope to remove them by introducing some additional simple factor χ_q that is also independent of p . Also, (4.22) is unchanged by multiplying $T(p, q)$ by any function η_p of p only.

We therefore try the ansatz

$$T_p(z_q, w_q) = \eta_p \chi_q \prod_{i=1}^3 R_1(q_i)^{a_i} R_3(\bar{q}_i)^{b_i} , \quad (6.4)$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are integers, to be determined. We first seek to satisfy the relations (4.22), and find that we can match the powers of $v_{pq} = v_1(p, q)v_3(p, q)$ therein by taking

$$a_2 = a_1 - 1 \quad , \quad a_3 = a_1 + 1 \quad , \quad b_2 = b_3 = 1 \quad , \quad b_1 = -2 \quad . \quad (6.5)$$

At this stage we are free to choose a_1 , which corresponds to multiplying $T_p(z_q, w_q)$ by a factor $R_1(q_1)R_1(q_2)R_1(q_3) = R_1(z_q)R_1(-1/w_q)R_1(-w_q/z_q)$. This factor has no effect on the relations (4.22) and is bounded, with no zeros or poles, for $p, q \in \mathcal{D}$. From the argument after (4.22), it must therefore be independent of q . Taking $t_q = \infty$ and $q_i = \omega$, or $t_q = 0$ and $q_i = \omega^2$, we obtain the identity

$$\prod_{j=1}^3 R_1(q_j) = R_1(\omega)^3 = R_1(\omega^2)^3 \quad (6.6)$$

for all q .

Without loss of generality we can therefore choose $a_1 = 0$, giving $a_2 = -1, a_3 = 1$. Substituting the ansatz (6.4) into (4.23), the R_1 and R_3 functions cancel out, leaving

$$\eta_p^3 \chi_q \chi_{Vq} \chi_{V^2q} = 1 \quad . \quad (6.7)$$

Since p and q are independent variables, η_p must be a constant (independent of both p and q). We can absorb this constant into the factor χ_q in (6.4), so we can set

$$\eta_p = 1 \quad . \quad (6.8)$$

We can calculate χ_q from (4.24). Take $i = 1$ therein, so $p' = A_1 p$ and $p'_j = x^{2-j}/p_j$ and $\bar{p}'_j = x^{1-3\delta(j,2)}/\bar{p}_j$ for $j = 1, 2, 3$. Let $R'_m(z)$ be the function $R_m(z)$ defined above, but with p replaced by p' . From the above definitions and properties we can verify that

$$R'_1(z)R_1(z) = \frac{p_1 p_3 g_1(z p_1) g_1(z/p_1)}{g_1(z p_3) g_1(z/p_3)}$$

and

$$R'_3(v)R_3(v) = \frac{1}{g_3(v/\bar{p}_2)g_3(x^2 v \bar{p}_2)} \quad . \quad (6.9)$$

Using these, we find (after some work) from (5.12) and (6.4) that (4.24) is satisfied for $i = 1$ iff

$$\chi_q^2 = \frac{g_3(\bar{q}_2)g_3(x^2 \bar{q}_2)g_3(\bar{q}_3)g_3(x^2 \bar{q}_3)}{g_3(\bar{q}_1)^2 g_3(x^2 \bar{q}_1)^2} \quad . \quad (6.10)$$

Taking $i = 2$ or $i = 3$ in (4.24) merely permutes p_1, p_2, p_3 in the working, which leaves the p -independent result (6.10) unchanged.

We can use (4.10) to eliminate the factors $g_3(\bar{q}_j)$ in favour of $g_3(x^2 \bar{q}_j)$. Using also (6.7), we obtain

$$\chi_q = \frac{g_3(x^2 \bar{q}_2)g_3(x^2 \bar{q}_3)}{\bar{q}_1 g_3(x^2 \bar{q}_1)^2} \quad . \quad (6.11)$$

Writing χ_q as $\chi(z_q, w_q)$ and using (4.10), we find that

$$\begin{aligned} -x \bar{q}_3 \chi(x^{-1}/z_q, x^{-1}/w_q) &= -x \bar{q}_2 \chi(x/z_q, 1/w_q) \\ &= x^{-2} \bar{q}_1^{-2} \chi(1/z_q, x/w_q) = \chi(z_q, w_q) . \end{aligned} \quad (6.12)$$

Our ansatz (6.4) is now

$$T_p(z_q, w_q) = \chi_q \frac{R_1(q_3) R_3(\bar{q}_2) R_3(\bar{q}_3)}{R_1(q_2) R_3(\bar{q}_1)^2} . \quad (6.13)$$

Using (6.3) and (6.12), we find that this expression does indeed satisfy the relations (4.22), so from the argument following (4.22) we see that the expression (6.13) must be correct to within a factor that is independent of q . It also satisfies (4.23), so this factor must be a cube root of unity. Since (4.24) is satisfied, this root must be unity itself. This completes the proof of (6.13).

(a) Relation to the order parameter function $\mathcal{S}(t_q)$

In an earlier paper (B 2006), the author considered the function $\mathcal{S}(t_q)$ that occurs in the derivation of the order parameter of the chiral Potts model. This is given by

$$\log S(t_q) = -\frac{2}{N^2} \log k + \frac{1}{2N\pi} \int_0^{2\pi} \frac{k' e^{i\theta}}{1 - k' e^{i\theta}} \log [\Delta(\theta) - t_q] d\theta , \quad (6.14)$$

$\Delta(\theta)$ being defined by (2.8). This is very similar to the function $\tau_2(p, q)$ of this paper, in fact if q_3 is the point (3) of eqn. (5.13), with $\mu_q^N = 1/k'$ and $t_q = 0$, and q_4 is the point (4), with $\mu_q = t_q = \infty$, then

$$S(t_q)^{2N} = \frac{\tau_2(p_3, q)}{k^{4/N} \tau_2(p_4, q)} . \quad (6.15)$$

For $N = 3$, we can use our results (4.19), (6.13) to express the RHS of (6.15) in terms of our elliptic-type functions. Because $\bar{q}_j = 1$ for both points (3) and (4), the R_3 factors cancel. The R_1 factors reinforce, giving

$$S(t_q)^3 = \frac{x_q G((-1/w_q)}{k^{1/3} G(z_q)} , \quad (6.16)$$

where

$$G(z) = F_1(\omega z) / F_1(\omega^2 z) .$$

This is indeed the result given in equations (57), (62) of (B 2006) (with p therein replaced by q). Because the R_3 functions have cancelled, it contains no elliptic-type functions with arguments z_q/w_q^2 , $-w_q/z_q^2$ or $-z_q w_q$.

7. Summary

For $N = 3$, we have written the integral expression (2.7) as a product of generalized elliptic functions, with arguments that are ratios and products of the variables $p_j, q_j, \bar{p}_j, \bar{q}_j$ defined in (4.5), (4.7). These are the variables of the alternative hyperelliptic parametrization of the chiral Potts model.

Is this progress? To the author it seems that the answer is indeed yes: functions such as the $F_r(z)$ of (6.1) occur naturally in the free energy of other models, such as the Ising, six and eight-vertex models, so it is interesting to see them occurring again in the three-state chiral Potts model. Indeed, it may indicate some intriguing relations between such models. Certainly it provides an explicit formulation of the meromorphic structure of $\tau_2(p, q)$ on its Riemann surface.

Unfortunately it is not clear how one could proceed to $N > 3$. There seems then to be no reason to expect to be able to express quantities such as v_{pq} as products of single-argument Jacobi elliptic functions. The best one can hope for is to write them as ratios of hyperelliptic theta functions, as in (B 1991b). These are entire functions of two or more related variables, all of whose zeros are simple. To write $\tau_2(p, q)$ in such a parametrization, one would need to generalize the hyperelliptic theta functions to make the order of the zeros increase linearly with the distance of the zero from some origin. The author knows of no such generalization. Indeed, while Jacobi's triple product identity enables one to write Jacobi theta functions as either products or sums, there seems no reason to expect the same of functions such as the numerators or denominators in (6.1).

There is also a problem with extending our working to the free energy of the full $N = 3$ chiral Potts, which is given by the double integral in eq. 46 of (B 1991a) and eq. 61 of (B 2003a). The author showed in (B 2003b) that the graph of the Riemann surface of this function has one more dimension than $\tau_2(p, q)$. If we fix p and consider the free energy as a function of q , then we do not have the relation (4.16) and we need a three-dimensional lattice to represent the Riemann surface, rather than the honeycomb lattice of Figure 3. This implies the need for one more “hyperelliptic” variable, in addition to z_q and w_q , and it quite unclear what this may be. (No such extra variable is needed for the $N = 2$ Ising case.)

Appendix A: The duality map

The mappings R, U, V, M leave k, k', x unchanged. There is another map that takes k' to $1/k'$ and p to p' , where

$$\{x_{p'}, y_{p'}, \mu_{p'}, t_{p'}\} = \{\omega^{1/4} x_p \mu_p, \omega^{1/4} y_p / \mu_p, 1/\mu_p, \omega^{1/2} t_p\} . \quad (\text{A1})$$

Let $x = e^{-\pi\lambda}$, $z = e^{i\pi\alpha}$ and write the RHS of (4.2) as $r(\lambda)$. Also, write $h(z), \phi(z)$ as $h(\alpha, \lambda), \phi(\alpha, \lambda)$. Set

$$\lambda' = 4\pi/(3\lambda) . \quad (\text{A2})$$

Then, as in §15.7 of (B 1980), one can establish the conjugate modulus relations

$$h(\alpha, \lambda) = \phi(-2i\alpha/\lambda, \lambda') \quad , \quad \phi(\alpha, \lambda) = h(2i\alpha/3\lambda, \lambda') \quad . \quad (\text{A3})$$

Also, $r(\lambda) = 1/r(\lambda')$, so replacing λ by λ' does indeed invert k' , as given by (4.2). Further, if $p_j = \exp(i\pi\alpha_j)$, then we can choose

$$\alpha_1 + \alpha_2 + \alpha_3 = -2 \quad .$$

(This enables one to take $\alpha_j = -2/3$, for all j , for the interesting case when $y_p = 0$ and $p_j = \omega^{-1}$.)

If we set

$$\alpha'_j = \frac{2i}{3\lambda}(\alpha_{j+1} - \alpha_{j-1} + i\lambda) \quad , \quad (\text{A4})$$

then it is also true that $\alpha'_1 + \alpha'_2 + \alpha'_3 = -2$. In fact the mapping $\lambda, \alpha_j \rightarrow \lambda', \alpha'_j$ is self-reciprocal. Further, using (A3), we find

$$h(\alpha_j, \lambda) = \phi(\alpha'_{j+1} - \alpha'_{j-1} + i\lambda', \lambda') \quad , \quad \phi(\alpha_{j+1} - \alpha_{j-1} + i\lambda, \lambda) = h(\alpha'_j, \lambda') \quad .$$

The left-hand sides of these last two equations are the functions $h(p_j)$, $\phi(x\overline{p}_j)$ of eqns. (4.9), (4.10). It follows that these two equations are interchanged by the duality mapping (A1), provided we also use the conjugate modulus form (A3) of the elliptic functions and then make the transformation $\lambda, \alpha_j \leftrightarrow \lambda', \alpha'_j$.

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